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On higher Fano varieties - a summary

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Abstract

A concise survey on higher Fano varieties is given. The focus is [Min18], where results of de Jong-Starr, Araujo-Castravet, and Suzuki are generalized.

- A gneral result for k-Fano manifolds.
- A general result which guarantees that any general point of X is contained in the image of a generaically injective morphism $R \rightarrow X$ from a rational k-fold R.

1 Background

Let us start with the following:

Campana, ASENS.1992 [Cam92]; Kollár-Miyaoka-Mori,JDG,1992 [KMM92]

Fano varieties (i.e., smooth complex projective varieties X with ample $-K_X$ ($\Longleftrightarrow$ $\mathrm{ch}_1(X) > 0$)) (e.g. smooth hypersurfaces of degree d in $\mathbb{P}^n$ $(d \leq n)$) are rationally connected.

With a glance at this result, topologists would wonder how this theorem could be generalized to "higher connected case", whatever that means. Then the origin of such To search for a clue, it is very natural to go back to the origin of the above theorem:

Shigefumi Mori, Ann.Math.1979 [Mor79]

Any Fano variety X of positive dimension is covered by rational curves (i.e. any general point $x \in X$ is contained in the image of a generically injective morphisms $\mathbb{P}^1 \rightarrow X$).

Since it is not obvious what kind of "higher connectivity" we should look after, in view of this fundamental theorem of Mori, we may instead look after the "covered by rational k-folds" property, i.e. any general point is contained in the image of a generaically injective morphism $\mathbb{P}^k \rightarrow X$ or more generally by rational projective k-folds instead of $\mathbb{P}^k$.

For this purpose, we should impose appropriate furhter restrictions of the Fano property, which would (under some mild assumption if necessary) guarantee the "covered by rational k-folds" property.

Now, it has become apparent that the following "k-Fano "property is very appropriate for this purpose, by the work of de Jong -Starr [dJS07] (for $k = 2$) and Araujo-Castravet [AC12] (for $k = 3$):

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k-Fano variety

Let us call a Fano variety X $\underline{k\text{-Fano}\ (k \geq 2)}$ when

$$\mathrm{ch}_i(X) > 0\ (1 \leq i \leq k-1),\ \mathrm{ch}_k(X) \geq 0, \tag{1}$$

and, for terminological convenience, we sometimes call an ordinary Fano variety <u>1-Fano.</u>

The basic strategy to cover a k-Fano X by rational k-folds is, starting with a k-Fano manifold X, under some extra condition, construct a sequence

$$X = H_0 \mapsto H_1 \mapsto \cdots H_{k-2} \mapsto H_{k-1},$$

where "$\mapsto$"means an inductive construction procedure explained below, but the points are:

- For $0 \leq i \leq k-1$, H_i is $k-i$-Fano;
- Since H_{k-1} is Fano, it is covered by a rational 1-fold by the Mori theorem. Inductively, if H_j is covered by a rational $k-j$-fold, show H_{j-1} is covered by a rational $k-j+1$-fold, under some condition if necessary. Thus, at the end, we see $X = H_0$ is covered by a rational k-fold.

Now the inductive construction "$\mapsto$"may be summarized the following itembox, which inputs a k-Fano manifold $X = H_0$ and outputs a $k-1$-Fano manifold $H_x = H_1$, where $x \in X$ is a general point (for the precise meanings of the terminologies and notations, consult [Kol96] [AC12]) :

$X \mapsto H_x$, which reflects "ample" information about X

- For a Fano manifold X of dimension n, with $x \in X$ a general point,

$$\begin{cases} \exists V_x & \underset{\text{irred. open}}{\subset} \quad \mathrm{Hom}(\mathbb{P}^1, X, 0 \mapsto x) \\ G_0 & := Aut\left(\mathbb{P}^1\right)_0 \ \subset \ Aut(\mathbb{P}^1) = PGL(2, \mathbb{C}) \end{cases}$$

$$\begin{tikzcd}
\mathbb{P}^1 \times V_x \arrow[r] \arrow[d] \arrow[rr, bend left, "(t,[f]) \mapsto f(t)"] & U_x = \left(\mathbf{P}^1 \times V_x\right)//G_0 \arrow[r, "\mathrm{ev}_x"] \arrow[d, "\pi_x"'] & X, \\
V_x \arrow[r] \arrow[u, bend left, "\{o\}\times\mathrm{id}"] & H_x := V_x//G_0 \arrow[u, bend right, "\sigma_x"'] &
\end{tikzcd} \tag{2}$$

- By results of Miyaoka [Kol96, V,3.7.5.Prop] and Kebekus [Keb02, Th.3.3], every curve parametrized by H_x is immersed at x, and the subvarietry $H_x^{\mathrm{Sing},x}$, parametrizing curves singular at x, is at most finite.

- There is a normalization onto its image (Kebekus [Keb02, Th.3.3,3.4] Hwang-Mok [HM04])

$$\tau_x : H_x \to \mathbb{P}\left(T_{X,x}\right) \cong \mathbb{P}^{n-1},$$

giving a polarization $(H_x, \tau_x^* \mathcal{O}(1)) =: (H_x, L_x)$, called a PMFRC (polarized minimal family of rational curves) through x.

- Denote this situation by $\boxed{X \mapsto H_x}$.

Here, the importance of the polarization L_x in PMFRC (H_x, L_x) becomes apparent with a glance at the following fundamental formula of Araujo-Castravet [AC12]:

ARAUJO-CASTRAVET, Prop.1.3 AJM 2012 [AC12]

- Let X be smooth complex projective uniruled,
- Let (H_x, L_x): PMFRC through a general point $x \in X$.

Then, for any $k \geq 1$, the following formula in the Chow ring with B_j : j-th Bernouilli number:

$$ch_k(H_x) = \sum_{j=0}^{k} \frac{(-1)^j B_j}{j!} c_1(L_x)^j \pi_{x*} \mathrm{ev}_x^* \big(ch_{k+1-j}(X)\big) - \frac{1}{k!} c_1(L_x)^k.$$

- Although a similar formula is expected for the Kontsevich's space, on which deJong-Starr [dJS07] based upon, it is yet to be worked out [MP17].

- The proof of Araujo-Castravet fundamental formula given in [?] is a tough computations (a transparent computational proof is given in [Min18]), exploiting the following description of the tangent bundle T_{H_x} by Druel [Dru06]:

A description of the tangent bundle T_{H_x} of H_x by Druel, Math. Ann. 2006 [Dru06]

$$
\begin{tikzcd}
\mathbb{P}^1 \times V_x \arrow[r] \arrow[d, shift left] \arrow[rr, bend left, "(t,[f])\mapsto f(t)"] & U_x = \left(\mathbf{P}^1 \times V_x\right)//G_0 \arrow[r, "\mathrm{ev}_x"] \arrow[d, "\pi_x"'] & X, \\
V_x \arrow[r] \arrow[u, bend left, "\{o\}\times\mathrm{id}"] & H_x := V_x//G_0 \arrow[u, bend right, "\sigma_x"']
\end{tikzcd}
$$

In the diagram (1), which quoted above, the section image $\sigma_x(H_x)$ is a divisor of U_x, whose corresponding line bundle on U_x is denoted by $\mathcal{O}_{U_x}(\sigma_x)$, or more simply, by (σ_x). Then, we have:

- $U_x \cong \mathbb{P}\big((\pi_x)_*\mathcal{O}_{U_x}(\sigma_x)\big)$.
- $T_{H_x} \cong (\pi_x)_*\Bigg(\big((ev_x^*T_X)/T_{\pi_x}\big)(-\sigma_x)\Bigg)$.

Now, Araujo-Castravet, under some extra conditions, proved their covering results by constructing the following sequences:

$$
\begin{aligned}
X \text{ (2-Fano)} &\mapsto H_x \text{ (1-Fano)} \\
X \text{ (3-Fano)} \mapsto H_x \text{ (2-Fano)} &\mapsto W_h \text{ (1-Fano)}
\end{aligned}
$$

by applying the fundamental formula at each "$\mapsto$. "

Thus, it is quite natural hope to iterate the Araujo-Castravet fundamental formula $k-1$-times to deal with the k-Fano case. Though, the computational complexty appears to explode out of our hands quickly.

However....

2 Suzuki's work [Suz16] and subsequent work [Min17][Nag18]

Taku Suzuki,[Suz16]

Let X be a weak k-Fano variety ($k \geq 2$) :

$$\mathrm{ch}_1(X) > 0,\ \mathrm{ch}_i(X) \geq 0\ (2 \leq i \leq k) \tag{3}$$

with $X \mapsto H_1$ such that

$$\dim H_1 \geq k^2 - k - 1,$$

And, either $2 \leq \boxed{k \leq 100}$ or some inequalities invloving Bernouilli numbers hold for $\leq k$. Then,

- $X \mapsto H_1$ is extended to a sequence $X \mapsto H_1 \mapsto \cdots \mapsto H_{k-1}$ with H_{k-1} a Fano of positive dimension.
- Suppose further
$$H_1 \ncong Q^{\dim H_1}, H_i \ncong \mathbb{P}^{\dim H_i}(1 \leq i < k),$$
then any general point of X is contained in the image of a generaically injective morphism $\mathbb{P}^k \to X$.

As I talk in Fukui [Min17], the above technical condition

$$\text{“either } 2 \leq \boxed{k \leq 100} \text{ or some inequalities invloving Bernouilli numbers hold for } \leq k \text{ ”} \tag{4}$$

can be removed. Now, just before the RIMS workshop for this Kokyuroku, I learnt Takahiro Nagooka also eliminated this technical condition around he same time and announced it in [Nag18].

In fact, both [Min17] and [Nag18], following the lead of [Suz16], start with explicit formulae of $g(i,k)_j$'s in

$$\mathrm{ch}_j(H_i) = \left(g(i,0)_j + \sum_{k=1}^{i} \underbrace{T^k\left(\mathrm{ch}_k(X)\right)}_{scalar} g(i,k)_j + \sum_{k=i+1}^{\min\{\dim X, i+j\}} \underbrace{T^i\left(\mathrm{ch}_k(X)\right) c_1(L_i)^{i-k}}_{\text{degree } 0} g(i,k)_j \right) c_1(L_i)^j,$$

where $T^i : A^*(X) \xrightarrow{T} A^*(H_1) \xrightarrow{T} \cdots \xrightarrow{T} A^*(H_i)$ with each T of the form $\pi_* \circ \mathrm{ev}^*$ (see (1) for the case $\pi = \pi_x, \mathrm{ev} = \mathrm{ev}_x$).

However, explicit formulae for $g(i,k)_j$'s given in [Min17] and [Nag18] are different; whereas [Nag18] express them using Bernouilli numbers, [Min17] expresses them as

$$g(i,k)_j = \begin{cases} -\frac{i}{j!} & k = 0 \\ \frac{(-1)^j k!}{j!} \sum_{q=\max\{k-i,1\}}^{j} \begin{bmatrix} i+q \\ k \end{bmatrix} \frac{1}{(i+q)!} \underbrace{\left(\sum_{l=1}^{q} (-1)^l \binom{q}{l} l^j \right)}_{=\ c(j,q)} & k \geq 1, j \geq \max\{k-i,1\} \\ 0 & k \geq 1, j < \max\{k-i,1\} \end{cases}$$

using the Stirling numbers of the first kind $\begin{bmatrix} p \\ k \end{bmatrix}$, which enjoys the following inductive characterizations:

$$\begin{cases} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 1, \quad \begin{bmatrix} m \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ n \end{bmatrix} = 0 \ (m \neq 0, n \neq 0) \\ \begin{bmatrix} n+1 \\ m \end{bmatrix} = \begin{bmatrix} n \\ m-1 \end{bmatrix} + n \begin{bmatrix} n \\ m \end{bmatrix} \end{cases}$$

Although it is nice to have eliminated the technical condition (4), we now realize the following problems, which are the motivation of this work:

Problems $\Longrightarrow$ Motivation

- For a complete intersection X of type $(d_1, \ldots, d_c)$ in $\mathbb{P}^n$,

  $$\mathrm{ch}_i(X) > 0 \ (resp.\ \mathrm{ch}_i(X) \geq 0) \iff \sum_{1 \leq j \leq c} d_j^i \leq n \ (resp. \sum_{1 \leq j \leq c} d_j^i \leq n+1),$$

  which implies:

  $$X \text{ is weak } k\text{-Fano (3)} \iff X \text{ is } k\text{-Fano (1)}$$

  Thus, we mould like to see how much we can gain by shifting from weak k-Fano to k-Fano.

- The final conclusin in Suzuki's formulation claims only the stronger condition that any general point of X is contained in the image of a generaically injective morphism $\mathbb{P}^k \to X$, under some (somewhat unsatisfactory) condition. So, we wonder what might be gained by searching for the weaker condition, replacing $\mathbb{P}^k$ with a general rational k-fold. In fact, the prototype of this phenomenon already happens for the 2-Fano case:

  A prototype

  Let X be 2-Fano with $\dim H_x \geq 1$, then the followings hold:

  [AC12] Any general point of X is contained in the image of a generaically injective morphism $\mathbb{P}^2 \to X$, provided $\mathrm{ch}_2(X) > 0$, and $(H_x, L_x) \ncong (\mathbb{P}^2, \mathcal{O}(2)), (\mathbb{P}^2, \mathcal{O}(3))$.

  [dJS07] Any general point of X is contained in the image of a generaically injective morphism $R \to X$ from a rational surface R, without any further assumption.

3 Main theorems

From our motivation just stated, I shall state our two main theorems of the following types:

- A gneral result for k-Fano manifolds.
- A general result which guarantees that any general point of X is contained in the image of a generaically injective morphism $R \to X$ from a rational k-fold R.

3.1 k-Fano version

Notice that the difference between the weak k-Fano (3) and the k-Fano (1) arises for $k \geq 3$, we now suppose $k \geq 3$. Then we main theorem reads as follows:

k-Fano version [Min18]

Let X be a $\underline{k\text{-Fano variety } (k \geq 3)}$:

$$\mathrm{ch}_i(X) > 0 \ (1 \leq i \leq k-1), \ \mathrm{ch}_k(X) \geq 0$$

with $X \mapsto H_1$ such that

$$\dim H_1 \geq k^2 - 2k - 1,$$

Then,

- $X \mapsto H_1$ is extended to a sequence $X \mapsto H_1 \mapsto \cdots \mapsto H_{k-1}$ with H_{k-1} a Fano of positive dimension.
- If $H_i \not\cong \mathbb{P}^{\dim H_i}$, or, if more generally

$$(H_i, L_i) \not\cong \left(\mathbb{P}^{\dim H_i}, \mathcal{O}(2)\right) (1 \leq i < k), \ (H_{k-1}, L_{k-1}) \not\cong \left(\mathbb{P}^1, \mathcal{O}(3)\right)$$

  then any generic point of X is contained in the image of a generically injective morphism $\mathbb{P}^k \to X$.

3.2 A general result which guarantees that any general point of X is contained in the image of a generaically injective morphism $R \to X$ from a rational k-fold R.

Here, our main theorem is the following:

weak version of covering by rational k-folds [Min18]

If we are given a sequence

$$X \mapsto H_1 \mapsto \cdots \mapsto H_{k-1}$$

with H_{k-1} a Fano of positive dimension, then any general point of X is contained in the image of a generaically injective morphism $R \to X$ from a rational k-fold R.

Our two main theorems stated above generalize the theorems of de Jong-Starr [dJS07], Araujo-Castravet [AC12] for $k = 2, 3$.

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